OSCILLATION OF SECOND ORDER LINEAR DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. In the present paper the problem of oscillation of all solutions of the second order linear delay equation

$$u''(t) + p(t)u(\tau(t)) = 0$$

is investigated, where p is a nonnegative locally summable function. For this equation a general oscillation criterion is obtained showing the joint contribution of the following two factors: the presence of the delay and the second order nature of the equation. Using this criterion, effective sufficient oscillation conditions are derived. Some of them concern delay equations only, and others involve ordinary differential equations as well . A number of known results, in particular a generalization of well-known Hille's criteria to delay equations, are improved. Several examples illustrate that some of the results obtained are best possible in a sense.

1. INTRODUCTION

Consider the linear second order delay equation

$$u''(t) + p(t)u(\tau(t)) = 0, \tag{1.1}$$

where $p : R_+ \to R_+$ is locally integrable, $\tau : R_+ \to R$ is continuous, $\tau(t) \le t$ for $t \ge 0$, $\tau(t) \to +\infty$ as $t \to +\infty$ and

$$\text{mes}\,\{s \ge t : p(s) > 0\} > 0 \qquad \text{for } t \ge 0, \tag{1.2}$$

where mes denotes the Lebesgue measure on the real line. These assumptions will be supposed to hold throughout the paper.

Let $T_0 = \min\{\tau(t) : t \ge 0\}$ and

$$\tau_{(-1)}(t) = \sup\{s \ge 0 : \tau(s) \le t\} \quad \text{for} \quad t \ge T_0.$$

Clearly $\tau_{(-1)}(t) \ge t$ for $t \ge T_0$, $\tau_{(-1)}$ is nondecreasing and coincides with the inverse of τ when the latter exists. Besides, put $\tau_{(-2)} = \tau_{(-1)} \circ \tau_{(-1)}$.

A continuous function $u : [t_0, +\infty[\to R$ is said to be a *solution* of (1.1) if it is locally absolutely continuous on $[\tau_{(-1)}(t_0), +\infty[$ along with its derivative and almost everywhere on $[\tau_{(-1)}(t_0), +\infty[$ satisfies (1.1). A solution of (1.1) is said to be *proper* if it is not identically zero in any neighbourhood of $+\infty$. A proper solution is called

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oscillatory (or it is said to oscillate) if it has a sequence of zeros tending to $+\infty$. Otherwise it is called *nonoscillatory.*

We say that the equation (1.1) is *oscillatory* if each one of its proper solutions oscillates. Otherwise we call (1.1) *nonoscillatory.*

The present paper is devoted to the problem of oscillation of (1.1). For the case of ordinary differential equations, i.e. when $\tau(t) \equiv t$, the history of the problem began as early as in 1836 by the work of Sturm [16] and was continued in 1893 by A. Kneser [11]. Essential contribution to the subject was made by E. Hille, A. Wintner, Ph. Hartman, W. Leighton, Z. Nehari, and others (see the monograph by C. Swanson [17] and the references cited therein). In particular, in 1948 E. Hille [6] obtained the following well-known oscillation criteria.

Let

$$\limsup_{t\to+\infty} t\int_t^{+\infty} p(s)ds > 1 \tag{1.3}$$

or

$$\liminf_{t\to+\infty} t\int_t^{+\infty} p(s)ds > \frac{1}{4}, \tag{1.4}$$

the conditions being assumed to be satisfied if the integral diverges. Then (1.1) *with* $\tau(t) \equiv t$ *is oscillatory.*

For the delay differential equation (1.1) earlier oscillation results can be found in the monographs by A. Myshkis [14] and S. Norkin [15]. In 1968 P. Waltman [19] and in 1970 J. Bradley [1] proved that (1.1) is oscillatory if $\int^{+\infty} p(t)dt = +\infty$. Proceeding in the direction of generalization of Hille's criteria, in 1971 J.Wong [21] showed that if $\tau(t) \geq \alpha t$ for $t \geq 0$ with $0 < \alpha \leq 1$, then the condition

$$\liminf_{t\to+\infty} t\int_t^{+\infty} p(s)ds > \frac{1}{4\alpha} \tag{1.5}$$

is sufficient for the oscillation of (1.1). In 1973 L. Erbe [2] generalized this condition to

$$\liminf_{t\to+\infty} t\int_t^{+\infty} \frac{\tau(s)}{s} p(s)ds > \frac{1}{4} \tag{1.6}$$

without any additional restriction on τ. In 1987 J. Yan [18] obtained some general criteria improving the previous ones.

An oscillation criterion of different type is given in 1986 by R. Koplatadze [7] and in 1988 by J. Wei [20], where it is proved that (1.1) is oscillatory if

$$\limsup_{t\to+\infty} \int_{\tau(t)}^t \tau(s)p(s)ds > 1 \tag{1.7}$$

or

$$\liminf_{t\to+\infty} \int_{\tau(t)}^t \tau(s)p(s)ds > \frac{1}{e}. \tag{1.8}$$

The conditions (1.7) and (1.8) are analogous to the oscillation conditions due to Ladas, Lakshmikantham and Papadakis [13], and Koplatadze and Chanturia [9], respectively,

$$L := \limsup_{t\to+\infty} \int_{\tau(t)}^{t} p(s)ds > 1, \tag{1.9}$$

$$l := \liminf_{t\to+\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e} \tag{1.10}$$

for the first order delay equation

$$u'(t) + p(t)u(\tau(t)) = 0. \tag{1.11}$$

The essential difference between (1.5)–(1.6) and (1.7)–(1.8) is that the first two can guarantee oscillation for ordinary differential equations as well, while the last two work only for delay equations. Unlike first order differential equations, where the oscillatory character is due to the delay only, the equation (1.1) can be oscillatory without any delay at all, i.e., in the case $\tau(t) \equiv t$. Figuratively speaking, two factors contribute to the oscillatory character of (1.1): the presence of the delay and the second order nature of the equation. The conditions (1.5)–(1.6) and (1.7)–(1.8) illustrate the role of these factors taken separately.

In the present paper, developing the ideas of [7], we obtain integral oscillation criteria for (1.1) where the joint contribution of the above mentioned factors is presented. These criteria are formulated in terms of solutions of certain integral inequalities and enable us to obtain new effective sufficient conditions for the oscillation of (1.1) generalizing (1.5)–(1.8) not only in the case of delay equations, but for ordinary differential equations as well. Several examples illustrate their worth.

In Section 2 a number of lemmas is given showing consecutive steps of our reasoning. Section 3 is dedicated to oscillation criteria caused by the presence of the delay. We show that these criteria have essentially first order character by reducing the problem of oscillation of (1.1) to that of a first order delay differential equation. In Section 4 we formulate a general oscillation theorem and some of its corollaries more convenient for obtaining effective sufficient conditions. In section 5 we obtain Hille type effective oscillation conditions for (1.1) which are due to its second order nature.

In what follows it will be assumed that the condition

$$\int^{+\infty} \tau(s)p(s)ds = +\infty \tag{1.12}$$

is fulfilled. As it follows from Lemma 4.1 in [8], this condition is necessary for (1.1) to be oscillatory. The paper being devoted to the problem of oscillation of (1.1), the condition (1.12) does not affect the generality.

2. Preliminary lemmas

Lemma 2.1. *Let* (1.12) *be fulfilled,* $u : [t_0, +\infty[\to]0, +\infty[$ *be a positive solution of* (1.1) *and* $T = \tau_{(-1)}(t_0)$. *Then*
(i) $u'(t) > 0,\ \ u(t) \ge t\,u'(t)$ *for* $t \ge T$;
(ii) u *is nondecreasing on* $[T, +\infty[$, *while the function* $t \mapsto u(t)/t$ *is nonincreasing*

on $[T, +\infty[$*;*
(iii) *for any function* $\nu : R_+ \to R$ *satisfying*

$$\nu(t) \le t \quad for \quad t \in R_+, \qquad \nu(t) \to +\infty \quad as \quad t \to +\infty, \tag{2.1}$$

we have

$$u(\tau(t)) \ge \tau_{/\nu}(t)\, u(\nu(t)) \quad for \quad t \ge \max\{T, \nu_{(-1)}(t_0)\}, \tag{2.2}$$

where

$$\tau_{/\nu}(t) = \begin{cases} 1 & if \ \tau(t) \ge \nu(t), \\ \frac{\tau(t)}{\nu(t)} & if \ \tau(t) \le \nu(t). \end{cases} \tag{2.3}$$

Proof. In view of (1.2) it is obvious that $u'(t) > 0$ for $t \ge 0$. Let $\rho(t) \equiv u(t) - t\,u'(t)$. Since $\rho'(t) = -t\,u''(t) \ge 0$ for $t \ge T$, we have either $u(t) - t\,u'(t) \ge 0$ for $t \ge T$ or $u(t) - t\,u'(t) < 0$ for $t \ge t_1$ with some $t_1 \ge T$. To prove (i), it suffices to show that the latter is impossible. Indeed, otherwise

$$\left(\frac{u(t)}{t}\right)' = \frac{t\,u'(t) - u(t)}{t^2} > 0 \quad \text{for} \quad t \ge t_1,$$

whence $u(\tau(t)) \ge c\,\tau(t)$ for $t \ge t_2 = \tau_{(-1)}(t_1)$ with some $c > 0$. The equation (1.1) then yields

$$u'(t_2) \ge \int_{t_2}^{+\infty} p(s)u(\tau(s))ds \ge c \int_{t_2}^{+\infty} p(s)\tau(s)ds$$

which contradicts (1.12). Thus (i) is proved. (ii) is an immediate consequence of (i), and (iii) follows from (ii). The proof is complete.

Remark 2.1. Without the condition (1.12) the following weaker versions of (i) and (iii) are valid (see [20], Lemma 1 and [2], Lemma 2.1, respectively): for each $0 < \gamma < 1$ there is $T_\gamma \ge T$ such that $u(t) \ge \gamma t\,u'(t)$ and $u(\tau(t)) \ge \gamma \tau_{/\nu}(t)\, u(\nu(t))$ for $t \ge T_\gamma$. It should be noted that in the applications below these versions would be sufficient.

Lemma 2.1 (i) implies

$$u(\tau(t)) \ge \tau(t)\, u'(\tau(t)) \quad \text{for} \quad t \ge T. \tag{2.4}$$

This inequality, however, can be improved.

Lemma 2.2. *Let* (1.12) *be fulfilled,* $u : [t_0, +\infty[\to]0, +\infty[$ *be a positive solution of* (1.1) *and* $T = \tau_{(-1)}(t_0)$*. Then*

$$u(\tau(t)) \ge \tau_T(t) u'(\tau(t)) \quad for \quad t \ge \tau_{(-1)}(T), \tag{2.5}$$

where

$$\tau_T(t) = \tau(t) + \int_T^{\tau(t)} \xi \tau(\xi) p(\xi) d\xi \quad for \quad t \ge \tau_{(-1)}(T). \tag{2.6}$$

Proof. Integrate the identity $(u(t) - t\,u'(t))' = t\,p(t)\,u(\tau(t))$ from T to $\tau(t) \geq T$ and use (2.5) to get

$$u(\tau(t)) \geq \tau(t)u'(\tau(t)) + \int_T^{\tau(t)} \xi p(\xi)u(\tau(\xi))d\xi \quad \text{for} \quad t \geq \tau_{(-1)}(T).$$

To estimate the last integral, use Lemma 2.1(iii) with $\nu(t) \equiv t$, Lemma 2.1 (i) and the nondecreasing character of u'. We get

$$\int_T^{\tau(t)} \xi p(\xi)u(\tau(\xi))d\xi \geq \int_T^{\tau(t)} \tau(\xi)p(\xi)u(\xi)d\xi \geq \int_T^{\tau(t)} \xi\tau(\xi)p(\xi)u'(\xi)d\xi \geq$$
$$\geq \left(\int_T^{\tau(t)} \xi\tau(\xi)p(\xi)d\xi\right)u'(\tau(t)) \quad \text{for} \quad t \in \tau_{(-1)}(T).$$

The last two inequalities imply (2.5). The proof is complete.

Lemma 2.2 immediately implies

Lemma 2.3. *Let* (1.12) *be fulfilled,* $u : [t_0, +\infty[\to]0, +\infty[$ *be a positive solution of* (1.1) *and* $T = \tau_{(-1)}(t_0)$. *Then the function* $x : [T, +\infty[\to]0, +\infty[$ *defined by* $x(t) = u'(t)$ *is a positive solution of the differential inequality*

$$x'(t) + \tau_T(t)x(\tau(t)) \leq 0, \tag{2.7}$$

where τ_T *is defined by* (2.6).

The estimate (2.5) is essential for the results of Section 3. Being more exact than (2.4), via Lemma 2.3 it will enable us to improve the criteria (1.7) and (1.8).

The following four lemmas are crucial in proving the general oscillation theorem in Section 4, especially Lemmas 2.5 and 2.7 giving important estimates. Note beforehand that a continuous function $v : [T, +\infty[\to]0, +\infty[$ ($w : [T, +\infty[\to]0, +\infty[$) is a solution of the integral inequality (2.8) (integral inequality (2.11)) if it satisfies (2.8) ((2.11)) for $t \geq \tau_{(-2)}(T)$ ($t \geq \nu_{(-1)}(T)$). The same is true for the integral equations (2.17) and (2.18). Note also that solutions of these integral inequalities and equations are necessarily positive.

Lemma 2.4. *Let* (1.12) *be fulfilled,* $u : [t_0, +\infty[\to]0, +\infty[$ *be a positive solution of* (1.1) *and* $T = \tau_{(-1)}(t_0)$. *Then the function* $v : [T, +\infty[\to]0, +\infty[$ *defined by* $v(t) = \frac{u'(\tau(t))}{u'(t)}$ *is a solution of the integral inequality*

$$v(t) \geq \exp\left\{\int_{\tau(t)}^{t} \tau_T(\xi)\,p(\xi)v(\xi)\,d\xi\right\}, \quad t \geq \tau_{(-2)}(T). \tag{2.8}$$

Proof. We have $v(t) = \frac{x(\tau(t))}{x(t)}$ for $t \geq T$, where, according to Lemma 2.3, x is a positive solution of (2.7). If we rewrite (2.7) as

$$\frac{x'(t)}{x(t)} \leq -\tau_T(t)\,p(t)\,v(t) \quad \text{for} \quad t \geq \tau_{(-1)}(T) \tag{2.9}$$

and integrate from t to $\tau(t)$, then we get (2.8) thus completing the proof.

Lemma 2.5. *Let* (1.12) *be fulfilled,* $u : [t_0, +\infty[\to]0, +\infty[$ *be a positive solution of* (1.1) *and* $T = \tau_{(-1)}(t_0)$. *Then there exists a solution* $v : [T, +\infty[\to]0, +\infty[$ *of* (2.8) *such that*

$$u'(s) \geq \exp\left\{\int_s^t \tau_T(\xi)p(\xi)\,v(\xi)\,d\xi\right\}u'(t) \quad \text{for} \quad t \geq s \geq \tau_{(-1)}(T). \tag{2.10}$$

Proof. By (2.9)

$$\frac{u''(t)}{u'(t)} \leq -\tau_T(t)p(t)v(t) \quad \text{for} \quad t \geq \tau_{(-1)}(T)$$

where v is a solution of the (2.8). Integrating this inequality from t to s, we get (2.10) thus completing the proof.

Lemma 2.6. *Let a continuous function* $\nu : R_+ \to R$ *satisfy* (2.1), $u : [t_0, +\infty[\to]0, +\infty[$ *be a positive solution of* (1.1) *and* $T = \tau_{(-1)}(t_0)$. *Then the function* $w : [T, +\infty[\to]0, +\infty[$ *defined by* $w(t) = \frac{u(\nu(t))}{u'(t)}$ *is a solution of the integral inequality*

$$w(t) \geq \int_T^{\nu(t)} \left\{\exp \int_s^t \tau_{/\nu}(\xi)p(\xi)w(\xi)\,d\xi\right\} ds, \quad t \geq \nu_{(-1)}(T), \tag{2.11}$$

where $\tau_{/\nu}$ *is defined by* (2.3).

Proof. If we write (1.1) as

$$(u'(t))' = -p(t)\frac{u(\tau(t))}{u'(t)}\,u'(t) \quad \text{for} \quad t \geq T, \tag{2.12}$$

then we have

$$u'(t) = u'(T)\exp\left\{-\int_T^t p(\xi)\,\frac{u(\tau(\xi))}{u'(\xi)}\,d\xi\right\}\,ds \quad \text{for} \quad t \geq T, \tag{2.13}$$

$$u(\nu(t)) \geq u'(T)\int_T^{\nu(t)} \exp\left\{-\int_T^s p(\xi)\,\frac{u(\tau(\xi))}{u'(\xi)}d\xi\right\}\,ds \quad \text{for}\ t \geq \nu_{-1}(T). \tag{2.14}$$

Dividing (2.14) by (2.13) and using (2.2), we get (2.11). The proof is complete.

Lemma 2.7. *Let a continuous function* $\nu : R_+ \to R$ *satisfy* (2.1), $u : [t_0, +\infty[\to]0, +\infty[$ *be a solution of* (1.1) *and* $T = \tau_{(-1)}(t_0)$. *Then there exists a solution* $w : [T, +\infty[\to]0, +\infty[$ *of* (2.11) *such that*

$$u(t) \geq \left(t + \int_T^t s\,\tau_{/\nu}(s)p(s)\,w(s)\,ds\right)u'(t) \quad \text{for} \quad t \geq T. \tag{2.15}$$

Proof. Integrate the identity $(u(t) - t\,u'(t))' = t\,p(t)\,u(\tau(t))$ from T to $t \geq T$ and use (2.2) to get

$$\begin{aligned} u(t) \ &\geq\ t\,u'(t) + \int_T^t s\,p(s)\frac{u(\tau(s))}{u'(s)}u'(s)\,ds \geq \\ &\geq\ \left(t + \int_T^t s\,\tau_{/\nu}(s)p(s)\,w(s)\,ds\right)u'(t) \ \text{ for } \ t \geq T, \end{aligned} \tag{2.16}$$

where, according to Lemma 2.6, w is a solution of (2.11). Thus (2.15) holds and the proof is complete.

Since (1.12) is necessary for the oscillation of (1.1), its violation via Lemmas 2.4 and 2.6 imply the existence of solutions of (2.8) and (2.11). The following two lemmas give more exact results which will permit us to do without the condition (1.12) in Section 4.

Lemma 2.8. *Let* (1.12) *be violated. Then the integral equation corresponding to* (2.8)

$$v(t) = \exp\left\{\int_{\tau(t)}^{t} \tau_T(s)\, p(s) v(s)\, ds\right\} \tag{2.17}$$

has a bounded solution.

Proof. Let $M > 1$ be an arbitrary number. There exists $\delta > 0$ such that $\exp(\delta M) \leq M$. Since (1.12) is violated, there exists $T_0 \geq 0$ such that $\int_{T_0}^{+\infty}(L+1)\tau(s)p(s)ds \leq \delta$, where $L = \int_0^{+\infty} \tau(s)p(s)ds$. We claim that for any $T \geq T_0$ (2.17) has a solution v satisfying $1 \leq v(t) \leq M$ for $t \geq T$. To show this, consider the bounded convex closed set $V = \{v \in C([T, +\infty[) : 1 \leq v(t) \leq M\}$ in the space $C([T, +\infty[)$ of all continuous on $[T, +\infty[$ functions with the topology of uniform convergence on every finite interval, and consider the operator Q on V defined by

$$Q(v)(t) = \begin{cases} \exp\left\{\int_{\tau(t)}^{t} \tau_T(s)\, p(s)\, v(s)\, ds\right\} & \text{for } t \geq \tau_{(-2)}(T), \\ Q(v)(\tau_{(-2)}(T)) & \text{for } T_0 \leq t \leq \tau_{(-2)}(T). \end{cases}$$

Since

$$\int_T^{\tau(s)} \xi\tau(\xi)p(\xi)d\xi \leq \tau(s) \int_0^{\tau(s)} \tau(\xi)p(\xi)d\xi \leq L\tau(s),$$

it can be easily checked that Q maps V into itself and satisfies all the conditions of the Schauder-Tychonoff fixed point theorem (see, e.g., [3], pp.161–163). The fixed point of Q obviously is a solution of (2.17). The proof is complete.

Lemma 2.9. *Let* (1.12) *be violated and a continuous function* $\nu : R_+ \longrightarrow R$ *satisfy* (2.1). *Then for all sufficiently large* T *the integral equation corresponding to* (2.11)

$$w(t) = \int_T^{\nu(t)} \exp\left\{\int_s^t \tau_{/\nu}(\xi)\, p(\xi)\, w(\xi)\, d\xi\right\} ds \tag{2.18}$$

has a solution w *such that* w/ν *is bounded.*

Proof. Let $M > 1$, $\delta > 0$ and $T_0 \geq 0$ be as in the proof of Lemma 2.8. Then for any $T \geq T_0$, (2.18) has a solution w satisfying $\nu(t) \leq w(t) \leq M\,\nu(t)$ for $t \geq T_0$. Indeed, using the inequality $\tau_{/\nu}(t)\,\nu(t) \leq \tau(t)$, we get convinced that the set $V = \{w \in C([T, +\infty[) : \nu(t) \leq w(t) \leq M\,\nu(t)\}$ and the operator Q defined on V by

$$Q(w)(t) = \begin{cases} \int_T^{\nu(t)} \exp\left\{\int_s^t \tau_{/\nu}(\xi)\, p(\xi)\, w(\xi)\, d\xi\right\} ds & \text{for } t \geq v_{(-1)}(T), \\ Q(w)(\tau_{(-1)}(T)) & \text{for } T \leq t \leq \nu_{(-1)}(T) \end{cases}$$

satisfy all the conditions of the Schauder-Tychonoff fixed point theorem. As above, the fixed point of Q is a solution of (2.18). The proof is complete.

3. Oscillations Caused by the Delay

In this section oscillation results are obtained for (1.1) by reducing it to a first order equation. Since for the latter the oscillation is due solely to the delay, the criteria hold for delay equations only and do not work in the ordinary case. The section is independent of the general oscillation Theorem 4.1 and is based only on Lemma 2.3. It should be observed, however, that by means of lower a priori asymptotic estimates for v (as in Section 5 for w) Theorems 3.3 and 3.5 (unlike Theorem 3.4) could be deduced from Corollary 4.2 below.

Lemma 2.3 immediately implies

Theorem 3.1. *Let* (1.12) *be fulfilled and the differential inequality* (2.7) *have no eventually positive solution. Then the equation* (1.1) *is oscillatory.*

Theorem 3.1 reduces the question of oscillation of (1.1) to that of the absence of eventually positive solutions of the differential inequality

$$x'(t) + \left(\tau(t) + \int_T^{\tau(t)} \xi\tau(\xi)p(\xi)d\xi\right) p(t)x(\tau(t)) \le 0. \tag{3.1}$$

So oscillation results for first order delay differential equations can be applied since the oscillation of the equation

$$u'(t) + g(t)u(\delta(t)) = 0 \tag{3.2}$$

is equivalent to the absence of eventually positive solutions of the inequality

$$u'(t) + g(t)u(\delta(t)) \le 0. \tag{3.3}$$

This fact is a simple consequence of the following comparison theorem deriving the oscillation of (3.2) from the oscillation of the equation

$$v'(t) + h(t)v(\sigma(t)) = 0. \tag{3.4}$$

We assume that $g, h : R_+ \to R_+$ are locally integrable, $\delta, \sigma : R_+ \to R$ are continuous, $\delta(t) \le t$, $\sigma(t) \le t$ for $t \in R_+$, and $\delta(t) \to +\infty$, $\sigma(t) \to +\infty$ as $t \to +\infty$.

Theorem 3.2. *Let*

$$g(t) \ge h(t) \ \ \textit{and} \ \ \delta(t) \le \sigma(t) \ \ \textit{for} \ \ t \in R_+, \tag{3.5}$$

and let the equation (3.4) *be oscillatory. Then* (3.2) *is also oscillatory.*

Corollary 3.1. *Let the equation* (3.2) *be oscillatory. Then the inequality* (3.3) *has no eventually positive solution.*

Proof. Suppose, to the contrary, that there exists a positive solution $u\colon [t_0, +\infty[\to R$ of (3.3). Then u is a solution of the equation $v'(t) + h(t)v(\delta(t)) = 0$, where $h(t) \equiv -\frac{u'(t)}{u(\delta(t))} \geq g(t)$. According to Theorem 3.2, the equation (3.2) must have a nonoscillatory solution which contradicts to the hypothesis of the corollary.

In the case $\delta(t) \equiv \sigma(t)$ Theorem 3.2 can be found in [5] (Theorem 3.1), and in the general case but under the additional restriction $\sigma(t) < t$ in [12] (Theorem 2.8). Since these restrictions are not imposed here, we present the proof, which, in our opinion, is interesting by itself.

Proof of Theorem 3.2. Let, to the contrary of the assertion of the theorem, (3.2) have a nonoscillatory solution $u : [t_0, +\infty[\to R$ which is supposed to be positive. In the space of all continuous on $[t_0, +\infty[$ functions with the topology of locally uniform convergence consider the set V consisting of all continuous $v : [t_0, +\infty[\to R$ satisfying

$$v(t) = u(t_0) \quad \text{for} \quad t_0 \leq t \leq T,$$

$$u(t) \leq v(t) \leq u(t_0) \quad \text{for} \quad t \geq T, \tag{3.6}$$

$$1 \leq \frac{v(\sigma(t))}{v(t)} \leq \frac{u(\delta(t))}{u(t)} \quad \text{for} \quad t \geq T, \tag{3.7}$$

where $T = \delta_{(-1)}(t_0)$. V is nonempty ($u \in V$) and bounded. Moreover, it is convex since

$$\frac{\lambda v_1(\sigma(t)) + (1-\lambda) v_2(\sigma(t))}{\lambda v_1(t) + (1-\lambda) v_2(t)} = \frac{\lambda}{v_2(t)} \left[\frac{\lambda}{v_2(t)} + \frac{1-\lambda}{v_1(t)}\right]^{-1} \frac{v_1(\sigma(t))}{v_1(t)} +$$

$$+ \frac{1-\lambda}{v_1(t)} \left[\frac{\lambda}{v_2(t)} + \frac{1-\lambda}{v_1(t)}\right]^{-1} \frac{v_2(\sigma(t))}{v_2(t)} \quad \text{for} \quad v_1,\ v_2 \in V,\ t \geq T.$$

Define the operator Q on V by

$$Q(v)(t) = \begin{cases} u(t_0) \exp\left\{-\int_{t_0}^t h(s) \frac{v(\sigma(s))}{v(s)} ds\right\} & \text{for} \quad t \geq T, \\ u(t_0) & \text{for} \quad t_0 \leq t \leq T. \end{cases}$$

Clearly $Q(v)(t) \leq u(t_0)$ for $t \geq t_0$. On the other hand, by (3.5) and (3.7) we get

$$Q(v)(t) \geq u(t_0) \exp\left\{-\int_{t_0}^t g(s) \frac{u(\delta(s))}{u(s)} ds\right\} = u(t) \quad \text{for} \quad t \geq T,$$

so (3.6) is fulfilled with Qv instead of v. The same is true for (3.7) since, by (3.5) and (3.7), we have

$$\begin{aligned} 1 \ &\leq \ \frac{Q(v)(\sigma(t))}{Q(v)(t)} = \exp\left\{\int_{\sigma(t)}^t h(s) \frac{v(\sigma(s))}{v(s)} ds\right\} \leq \\ &\leq \ \exp\left\{\int_{\delta(t)}^t g(s) \frac{u(\delta(s))}{u(s)} ds\right\} = \frac{u(\delta(t))}{u(t)} \quad \text{for} \quad t \geq T. \end{aligned}$$

Thus $QV \subset V$. Besides, standard arguments show that T is completely continuous in the topology of uniform convergence on every finite segment. Hence the Schauder-Tychonoff fixed point theorem implies the existence of v_0 such that $Qv_0 = v_0$ which obviously is a nonoscilatory solution of (3.4). The obtained contradiction proves the theorem.

Turning to applications of Theorem 3.1, we will use it together with the criteria (1.9) and (1.10) to get

Theorem 3.3. *Let*

$$K := \limsup_{t\to+\infty} \int_{\tau(t)}^{t} \left(\tau(s) + \int_0^{\tau(s)} \xi\tau(\xi)p(\xi)d\xi\right) p(s)ds > 1 \tag{3.8}$$

or

$$k := \liminf_{t\to+\infty} \int_{\tau(t)}^{t} \left(\tau(s) + \int_0^{\tau(s)} \xi\tau(\xi)p(\xi)d\xi\right) p(s)ds > \frac{1}{e}. \tag{3.9}$$

Then the equation (1.1) *is oscillatory.*

To apply Theorem 3.1, it suffices to note that: (i) (1.12) is fulfilled since otherwise $k = K = 0$; (ii) since $\tau(t) \to +\infty$ as $t \to +\infty$, the relations (3.8)-(3.9) imply the same relations with 0 changed by any $T \geq 0$.

Remark 3.1. Theorem 3.3 improves the criteria (1.7)–(1.8) of R. Koplatadze [7] and J. Wei [20] mentioned in the introduction. This is directly seen from (3.8)–(3.9) and can be easily checked if we take $\tau(t) \equiv t - \tau_0$ and $p(t) \equiv p_0/(t-\tau_0)$ for $t \geq 2\tau_0$, where the constants $\tau_0 > 0$ and $p_0 > 0$ satisfy $\tau_0 p_0 < 1/e$. In this case neither of (1.7)–(1.8) is applicable for (1.1) while both (3.8)–(3.9) give the positive conclusion about its oscillation. Note also that this is exactly the case where the oscillation is due to the delay since the corresponding equation without delay is nonoscillatory.

Remark 3.2. The criteria (3.8)–(3.9) look like (1.9)–(1.10), but there is an essential difference between them pointed out in the introduction. The condition (1.10) is close to the necessary one since according to [9] if $L \leq 1/e$, then (3.2) is nonoscillatory. On the other hand, for an oscillatory (1.1) without delay we have $k = K = 0$. Nevertheless, the constant $1/e$ in Theorem 3.3 is also best possible in the sense that for any $\varepsilon \in]0, 1/e]$ it can not be replaced by $1/e - \varepsilon$ without affecting the validity of the theorem. This is illustrated by the following

Example 3.1. Let $\varepsilon \in]0, 1/e]$, $1 - e\varepsilon < \beta < 1$, $\tau(t) \equiv \alpha t$ and $p(t) \equiv \beta(1-\beta)\alpha^{-\beta}t^{-2}$, where $\alpha = e^{\frac{1}{\beta-1}}$. Then (3.9) is fulfilled with $1/e$ replaced by $1/e - \varepsilon$. Nevertheless (1.1) has a nonoscillatory solution, namely $u(t) \equiv t^\beta$. Indeed, denoting $c = \beta(1-\beta)\alpha^{-\beta}$, we see that the expression under the limit sign in (3.9) is constant and equals $\alpha c|\ln\alpha|\,(1+\alpha c) = (\beta/e)\,(1 + (\beta(1-\beta))\,/e) > \beta/e > 1/e - \varepsilon$.

There is a gap between the conditions (1.9)–(1.10) and (3.8)–(3.9) when $0 \leq l \leq 1/e$, $l < L$, and $0 \leq k \leq 1/e$, $k < K$, respectively. In the case of first order equations there arises an interesting problem of filling this gap, i.e. of finding of a function $f : [0, 1/e] \longrightarrow [1/e, 1]$ such that the condition $L > f(l)$ would guarantee the oscillation of (3.2). Moreover, it makes sense to seek for an optimal function in the sense that $L < f(l)$ would imply nonoscillation. A number of papers are devoted to this problem (see, for example, [4] and the references therein). Using results in this direction, one can derive various sufficient conditions for the oscillation of (1.1).

According to Remark 3.1, neither of them can be optimal in the above sense but nevertheless they are of interest since they cannot be derived from Corollary 4.2 of the general oscillation theorem. We combine Theorem 3.1 with the best to our knowledge result in this direction ([4], Corollary 1) to obtain

Theorem 3.4. *Let K and k be defined by* (3.8)–(3.9), $0 \le k \le 1/e$ *and*

$$K > k + \frac{1}{\lambda(k)} - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2},$$

where $\lambda(k)$ is the smaller root of the equation

$$\lambda = \exp(k\lambda). \tag{3.10}$$

Then (1.1) *is oscillatory.*

Finally we give a criterion which follows from Theorem 3.1 and a simplified version of Theorem 3 in [10]. For the sake of simplicity we will formulate the theorem in terms of τ_0 (see (2.6)).

Theorem 3.5. *Let k be defined by* (3.9), $0 \le k \le 1/e$ *and*

$$\limsup_{t \to +\infty} \int_{\delta(t)}^{t} p(s)\tau_0(s) \exp\left(\lambda(k) \int_{\delta(s)}^{\delta(t)} p(\xi)\tau_0(\xi) d\xi\right) ds > 1,$$

where $\lambda(k)$ is the smaller root of the equation (3.10). *Then* (1.1) *is oscillatory.*

4. General Oscillation Criteria

In this section we prove a general oscillation theorem for (1.1). We first mention two criteria which are immediate consequences of Lemmas 2.4 and 2.6, respectively.

Proposition 4.1. *Let* (1.12) *be fulfilled and the integral inequality* (2.8) *have no solution. Then the equation* (1.1) *is oscillatory.*

Proposition 4.2. *Let* (1.12) *be fulfilled and there exist a continuous function $\nu : R_+ \to R$ satisfying* (2.1) *and such that for any $T \ge \nu_{(-1)}(0)$ the integral inequality* (2.11) *has no solution. Then the equation* (1.1) *is oscillatory.*

Now we formulate our main result.

Theorem 4.1. *Let there exist continuous functions $\nu, \sigma, \delta : R_+ \to R$ such that σ, δ are nondecreasing,*

$$\begin{gathered} \nu(t) \le t, \ \tau(t) \le \delta(t) \le t, \ 0 < \sigma(t) \le \delta(t) \quad \text{for} \quad t \ge 0, \\ \nu(t), \sigma(t) \to +\infty \quad \text{as} \quad t \to +\infty, \end{gathered} \tag{4.1}$$

and for any $T \geq \tau_{(-1)}(0)$, *any positive solution* v *of* (2.8) *and any positive solution* w *of* (2.11) *the inequality*

$$\limsup_{t\to\infty}\left\{\int_{\delta(t)}^{t} p(s)\left(\tau(s)+\int_{T}^{\tau(s)} \xi\tau_{/\nu}(\xi)\,p(\xi)w(\xi)d\xi\right)\times \right. \tag{4.2}$$
$$\times\exp\left(\int_{\delta(s)}^{\delta(t)} \tau_{T}(\xi)p(\xi)v(\xi)d\xi\right)ds+$$
$$\left.+\left(\sigma(t)+\int_{T}^{\sigma(t)} s\tau_{/\nu}(\xi)\,p(\xi)w(\xi)d\xi\right)\int_{t}^{+\infty}\tau_{/\sigma}(\xi)\,p(\xi)\,d\xi\right\}>1$$

holds, where τ_T *is defined by* (2.6) *and* $\tau_{/\nu}, \tau_{/\sigma}$ *by* (2.3). *Then the equation* (1.1) *is oscillatory.*

Proof. First of all note that the condition (4.2) implies (1.12). Indeed, suppose that (1.12) is violated. Then by Lemmas 2.8 and 2.9 the integral equations (2.17) and (2.18) have solutions v_0 and w_0, respectively, such that $v_0(t) \leq M$ and $w_0(t) \leq M\,\tau(t)$ with some $M > 1$. Using these inequalities along with the negation of (1.12), one can easily see that for $v \equiv v_0$ and $w \equiv w_0$ the left-hand side of (4.2) is zero. This proves that (1.12) holds.

Suppose now that, contrary to the assertion of the theorem, the equation (1.1) has a nonoscillatory solution $u : [t_0, +\infty[\to R$ which we may and will assume to be positive. Put $T = \tau_{(-1)}(t_0)$. By Lemma 2.1

$$u(\tau(t)) \geq \tau_{/\sigma}(t)u(\sigma(t)) \quad \text{for} \quad t \geq \max\{T, \sigma_{(-1)}(t_0)\}. \tag{4.3}$$

On the other hand, according to Lemmas 2.5 and 2.7 and because u' is nonincreasing, there exist positive solutions v and w of the integral inequalities (2.8) and (2.11), respectively, such that

$$u'(\tau(s)) \geq u'(\delta(s)) \geq E(v)(s,t)u'(\delta(t)) \quad \text{for} \quad t \geq s \geq \tau_{(-2)}(T), \tag{4.4}$$

$$u(\tau(s)) \geq F_\tau(w)(s)\,u'(\tau(s)) \quad \text{for} \quad s \geq \tau_{(-1)}(T), \tag{4.5}$$

$$u(\sigma(t)) \geq F_\sigma(w)(t)u'(\sigma(t)) \;\geq F_\sigma(w)(t)u'(\delta(t)) \quad \text{for} \quad t \geq \sigma_{-1}(T), \tag{4.6}$$

where for any $\mu : R_+ \longrightarrow R$ we set

$$E(v)(s,t) \;=\; \exp\left(\int_{\delta(s)}^{\delta(t)} \tau_T(\xi)\,p(\xi)\,v(\xi)\,d\xi\right),$$
$$F_\mu(w)(t) \;=\; \left(\mu(t)+\int_{T}^{\mu(t)} \xi\tau_{/\nu}(\xi)\,p(\xi)\,w(\xi)\,d\xi\right).$$

Integrating (1.1) from $\delta(t)$ to $+\infty$ and taking into account (4.1) and (4.3)–(4.6) along with the nondecreasing character of u, σ, and δ, we get

$$u'(\delta(t)) \geq \int_{\delta(t)}^{t} p(s)\, u(\tau(s))\, ds + \int_{t}^{+\infty} p(s)\, u(\tau(s))\, ds \geq$$

$$\geq \int_{\delta(t)}^{t} p(s) F_\tau(w)(s) u'(\tau(s))\, ds + u(\sigma(t)) \int_{t}^{+\infty} \tau_{/\sigma}(s)\, p(s)\, ds \geq$$

$$\geq u'(\delta(t)) \left\{ \int_{\delta(t)}^{t} p(s) F_\tau(w)(s) E(v)(s,t) ds + F_\sigma(w)(t) \int_{t}^{+\infty} \tau_{/\sigma}(s) p(s)\, ds \right\}$$

for large t. But this contradicts (4.2). The proof is complete.

Remark 4.1. Propositions 4.1 and 4.2 can be considered as included in Theorem 4.1 by assuming formally that if there are no such v and w, then (4.2) is automatically fulfilled.

Theorem 4.1 and its corollaries below enable one to obtain effective sufficient conditions for the oscillation of (1.1) by means of a priori asymptotic lower estimates for v and w (or by means of establishing of nonexistence of v or w which in a way may be considered as the existence of a lower estimate identically equal to $+\infty$). We will derive nontrivial estimates of this type in Section 5.

Now we formulate some corollaries of the theorem. We begin with one which shows the joint effect of the delay and the second order nature of (1.1) in its simplest form.

Corollary 4.1. *Let τ be nondecreasing and*

$$\limsup_{t\to\infty} \left\{ \int_{\tau(t)}^{t} p(s)\tau(s) ds + \tau(t) \int_{t}^{+\infty} p(s)\, ds \right\} > 1.$$

Then the equation (1.1) *is oscillatory.*

Taking the first term in (4.2) with $\nu(t) \equiv t$ and using the obvious estimate $w(t) \geq t - T$, we obtain

Corollary 4.2. *Let there exist a nondecreasing function $\delta : R_+ \to R$ satisfying $\tau(t) \leq \delta(t) \leq t$ for $t \geq 0$ and such that for any solution v of* (2.8) *the inequality*

$$\limsup_{t\to\infty} \left\{ \int_{\delta(t)}^{t} p(s)\tau_0(s) \exp\left(\int_{\delta(s)}^{\delta(t)} \tau_0(\xi)p(\xi)v(\xi)d\xi \right) ds \right\} > 1$$

holds, where τ_0 is defined by (2.6). *Then the equation* (1.1) *is oscillatory.*

Corollary 4.2 shows the contribution of the delay to the oscillation of (1.1). As it has been pointed out in Section 3, some of (but not all) the results of that section could be derived from it.

Analogously, taking the second term in (4.2) with $\nu(t) \equiv t$ and using the estimate $w(t) \geq t - T$, we obtain

Corollary 4.3. *Let there exist a nondecreasing function* $\sigma : R_+ \to R$ *satisfying* $\sigma(t) \le \tau(t) \le t$ *for* $t \ge 0$, $\lim_{t\to+\infty} \sigma(t) = +\infty$ *and such that the inequality*

$$\limsup_{t\to\infty} \left\{ \left(\sigma(t) + \int_0^{\sigma(t)} s\tau(s)\, p(s) ds \right) \int_t^{+\infty} p(s)\, ds \right\} > 1$$

holds. Then the equation (1.1) *is oscillatory.*

In the case of ordinary differential equations Corollary 4.3 implies the following test.

Corollary 4.4. *If*

$$\limsup_{t\to\infty} \left\{ \left(t + \int_0^t s^2\, p(s) ds \right) \int_t^{+\infty} p(s)\, ds \right\} > 1, \tag{4.7}$$

then the equation

$$u''(t) + p(t)u(t) = 0 \tag{4.8}$$

is oscillatory.

Corollary 4.4 yields the following improvement of Hille's criteria (1.3) and (1.4) in the class of functions p satisfying

$$p(t) \ge \frac{c_0}{t^2} \quad \text{for large} \quad t. \tag{4.9}$$

Corollary 4.5. *Let* (4.9) *be fulfilled with* $c_0 \in]0, \frac{1}{4}]$ *and*

$$\limsup_{t\to+\infty} t \int_t^{+\infty} p(s) ds > \frac{1}{1+c_0}.$$

Then (4.8) *is oscillatory.*

The condition (4.7) improves Hille's criteria even in the case where $c_0 = 0$. This is illustrated by the following

Example 4.1. Let the sequences of real numbers $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ be such that $a_k < b_k < a_{k+1}$ for $k = 1, 2, \dots$, $a_k \uparrow +\infty$ and $b_k \uparrow +\infty$ as $k \to \infty$, and

$$\lim_{k\to\infty} \frac{a_k}{b_k} = 0, \qquad \lim_{k\to\infty} \frac{b_k}{a_{k+1}} = 0$$

(for instance, we can take $a_k = 2^{k^2}$, $b_k = \frac{2^{k^2} + 2^{(k+1)^2}}{2}$). Let $\delta \in]0, \frac{3-\sqrt{5}}{2}[$ and $\varepsilon \in]0, 1[$ be such that $(1-\delta)(2-\delta)(1-\varepsilon) > 1$. Then for the function p defined by

$$p(t) = \begin{cases} \frac{1-\delta}{t^2} & \text{for } \ t \in]a_k, b_k[\\ 0 & \text{for } \ t \in]b_k, a_{k+1}[\end{cases}, \qquad k = 1, 2, \dots,$$

both conditions (1.3) and (1.4) are violated while (4.7) is fulfilled. This means that Corollary 4.4 gives a positive answer to the question of oscillation of the equation (4.8) even in the case where both Hille criteria fail.

Indeed, we have

$$\limsup_{t\to+\infty} t\int_t^{+\infty} p(s)ds \le \limsup_{t\to+\infty} t\int_t^{+\infty} \frac{1-\delta}{s^2}ds = 1-\delta < 1$$

and

$$\liminf_{t\to+\infty} t\int_t^{+\infty} p(s)ds \le \lim_{k\to\infty} b_k\int_{b_k}^{+\infty} \frac{1-\delta}{s^2}ds = \lim_{k\to\infty} b_k\int_{a_{k+1}}^{+\infty} \frac{1-\delta}{s^2}ds =$$
$$= \lim_{k\to\infty} \frac{(1-\delta)b_k}{a_{k+1}} = 0.$$

On the other hand, denoting $a_k^* = a_k+\varepsilon(b_k-a_k)$, we have $a_k/a_k^* \to 0$ and $a_k^*/b_k \to \varepsilon$ as $k \to \infty$, so that

$$\limsup_{t\to+\infty}\left(t+\int_0^t s^2\,p(s)ds\right)\int_t^{+\infty} p(s)\,ds \ge$$
$$\ge \limsup_{k\to\infty}\left(a_k^*+\int_{a_k}^{a_k^*}(1-\delta)\,ds\right)\int_{a_k^*}^{b_k}\frac{1-\delta}{s^2}\,ds \ge$$
$$\ge \limsup_{k\to\infty}\left(1+(1-\delta)\left(1-\frac{a_k}{a_k^*}\right)\right)(1-\delta)\left(1-\frac{a_k^*}{b_k}\right) \ge$$
$$\ge (2-\delta)\,(1-\delta)\,(1-\varepsilon) > 1.$$

The following corollary will be used in Section 5 (we take $\delta(t) \equiv t$ and $\sigma = \nu$).

Corollary 4.6. *Let there exist a nondecreasing continuous function $\nu : R_+ \longrightarrow R$ such that $0 < \nu(t) \le \tau(t) \le t$, $\nu(t) \to +\infty$ as $t \to +\infty$ and for any $T \ge \nu_{(-1)}(0)$ and any solution w of (2.11) the inequality*

$$\limsup_{t\to\infty}\left(\nu(t)+\int_T^{\nu(t)} s\tau_{/\nu}(s)\,p(s)w(s)ds\right)\int_t^{+\infty}\tau_{/\nu}(s)\,p(s)\,ds > 1$$

holds, where $\tau_{/\nu}$ is defined by (2.3). Then the equation (1.1) is oscillatory.

Corollary 4.6, like Corollary 4.3, exhibits the role of the factors not depending on the presence of the delay. Next section is devoted to this topic.

5. Oscillations Due to the Second Order Nature of the Equation

In this section, using Corollary 4.6, we will derive oscillation criteria for (1.1) which are due to the second order nature of the equation. They generalize Hille's criterion (1.4) to delay equations.

Theorem 5.1. *Let $\alpha \in]0,1], \tau(t) \ge \alpha t$ for large t and*

$$\liminf_{t\to+\infty} t\int_t^{+\infty} p(s)\,ds > c(\alpha), \tag{5.1}$$

where

$$c(\alpha) = \max\{\alpha^{\lambda-1}\lambda(1-\lambda) : 0 \le \lambda \le 1\}. \tag{5.2}$$

Then (1.1) is oscillatory.

Proof. Let $T \geq 0$, $\nu(t) \equiv \alpha t$, so that $\tau_{/\nu} \equiv 1$, and w be a solution of (2.11). By Corollary 4.6, it suffices to prove that the inequality

$$\limsup_{t\to+\infty}\left(\alpha t + \int_T^{\alpha t} sp(s)w(s)\,ds\right)\int_t^{+\infty} p(s)\,ds > 1 \tag{5.3}$$

holds. This is the case if $\limsup\limits_{t\to+\infty} \alpha t \int_t^{+\infty} p(s)\,ds > 1$, so we can suppose that

$$t\int_t^{+\infty} p(s)\,ds \leq 1/\alpha \quad \text{for large } t. \tag{5.4}$$

Put

$$\lambda_* = \liminf_{t\to+\infty} w(t)\left(\int_t^{+\infty} p(s)\,ds\right). \tag{5.5}$$

From (2.11) it is clear that $w(t) \geq \alpha t - T$, so $\lambda_* > \alpha c(\alpha) > 0$. We claim that $\lambda_* > 1$. Indeed, suppose to the contrary that $\lambda_* \in]0,1]$ and take $c_0 \in]\,c(\alpha), \liminf\limits_{t\to+\infty} t\int_t^{+\infty} p(s)\,ds\,[$. By (5.1) and (5.5) for any $\lambda \in]0,\lambda_*[$ there is $t_0 \geq T$ such that

$$w(t)\left(\int_t^{+\infty} p(s)\,ds\right) \geq \lambda, \quad t\int_t^{+\infty} p(s)\,ds \geq c_0 \quad \text{for } t \geq t_0. \tag{5.6}$$

Hence by (2.11) we have for $t \geq t_0/\alpha$

$$\begin{aligned} w(t) &\geq \int_{t_0}^{\alpha t} \exp\left\{\lambda \int_s^t p(\xi)\left(\int_\xi^{+\infty} p(\zeta)\,d\zeta\right)^{-1} d\xi\right\} ds = \\ &= \int_{t_0}^{\alpha t} \exp\left\{\lambda \ln \frac{\int_s^{+\infty} p(\zeta)\,d\zeta}{\int_t^{+\infty} p(\zeta)\,d\zeta}\right\} ds = \\ &= \left(\int_t^{+\infty} p(\zeta)\,d\zeta\right)^{-\lambda} \int_{t_0}^{\alpha t} s^{-\lambda}\left(s\int_s^{+\infty} p(\zeta)\,d\zeta\right)^{\lambda} ds \geq \\ &\geq \left(\int_t^{+\infty} p(\zeta)\,d\zeta\right)^{-\lambda} c_0^{\lambda}\,\frac{(\alpha t)^{1-\lambda} - t_0^{1-\lambda}}{1-\lambda}. \end{aligned} \tag{5.7}$$

Therefore by (5.6)

$$\begin{aligned} w(t)\int_t^{+\infty} p(\zeta)\,d\zeta &\geq \left(\int_t^{+\infty} p(\zeta)\,d\zeta\right)^{1-\lambda} \frac{(\alpha t)^{1-\lambda} c_0^{\lambda}}{1-\lambda} + o(1) = \\ &= \left(t\int_t^{+\infty} p(\zeta)\,d\zeta\right)^{1-\lambda} \frac{\alpha^{1-\lambda} c_0^{\lambda}}{1-\lambda} + o(1) \geq \frac{\alpha^{1-\lambda} c_0}{1-\lambda} + o(1). \end{aligned}$$

Passing here to lower limit, we get

$$\lambda_* \geq \frac{\alpha^{1-\lambda} c_0}{1-\lambda}.$$

Since $\lambda \in]0,\lambda_*]$ was arbitrary, we have

$$\alpha^{\lambda_*-1}\lambda_*(1-\lambda_*) \geq c_0 > c(\alpha), \tag{5.8}$$

which contradicts (5.2). The obtained contradiction shows that $\lambda_* > 1$. Therefore (5.7) for any $\lambda \in]1, \lambda_*]$ yields

$$w(t)\int_t^{+\infty} p(\zeta)\,d\zeta \geq \frac{c_0^\lambda}{\lambda-1}\left(\int_t^{+\infty} p(\zeta)\,d\zeta\right)^{1-\lambda}\left[t_0^{1-\lambda}-(\alpha\,t)^{1-\lambda}\right]$$

which tends to $+\infty$ as $t \to +\infty$. This means that $\lambda_* = +\infty$ and so in the last inequality λ can be any number from $]1, +\infty[$. Rewrite this inequality as

$$w(t) \geq \frac{c_0^\lambda\, t^\lambda}{\lambda-1}\left(t\int_t^{+\infty} p(\zeta)\,d\zeta\right)^{-\lambda}\left[t_0^{1-\lambda}-(\alpha\,t)^{1-\lambda}\right].$$

Hence, in view of (5.4), it follows the existence of $M > 0$ and $t_1 \geq t_0$ such that

$$w(t) \geq M\,t^\lambda \quad \text{for} \quad t \geq t_1,$$

i.e., for any $\lambda > 1$

$$w(t) \geq t^\lambda \quad \text{for large} \quad t. \tag{5.9}$$

Using (5.9) for $\lambda = 2$ along with (5.2) and (5.4), and integrating by parts, we get for large t

$$\begin{aligned}\int_T^{\alpha\,t} sp(s)w(s)\,ds \geq \int_{t^{1/2}}^{\alpha\,t} s^3p(s)\,ds \geq\\ \geq -t\int_{t^{1/2}}^{\alpha\,t} s\,d\left(\int_s^{+\infty} p(\xi)\,d\xi\right) =\\ = t\left(t^{1/2}\int_{t^{1/2}}^{+\infty} p(\xi)\,d\xi - \alpha t\int_{\alpha\,t}^{+\infty} p(\xi)\,d\xi + \int_{t^{1/2}}^{\alpha\,t}\left(\int_s^{+\infty} p(\xi)\,d\xi\right)ds\right) \geq\\ \geq t\left(-\frac{1}{\alpha} + \int_{t^{1/2}}^{\alpha\,t}\frac{c(\alpha)}{s}ds\right) = t\left(-\frac{1}{\alpha} + c(\alpha)\ln\alpha + \frac{c(\alpha)}{2}\ln t\right).\end{aligned} \tag{5.10}$$

Hence, in view of (5.1), we have (5.3). The proof is complete.

Remark 5.1. The constant $c(\alpha)$ is best possible in the sense that in (5.1) the strict inequality cannot be replaced by the nonstrict one without affecting the validity of the theorem. Indeed, denoting by λ_0 the point where the maximum in (5.2) is attained, we can see that the function $u(t) \equiv t^{1-\lambda_0}$ is a nonoscillatory solution of the equation $u''(t) + (c(\alpha)/t^2)u(\alpha t) = 0$.

Remark 5.2. We have $\alpha c(\alpha) = \max\{\alpha^\lambda\lambda(1-\lambda) : 0 \leq \lambda \leq 1\} < \max\{\lambda(1-\lambda) : 0 \leq \lambda \leq 1\} = 1/4$ for $0 < \alpha < 1$. Therefore for any $\alpha \in]0, 1[$ Theorem 5.1 improves the result of Wong (1.5).

Remark 5.3. Using Corollary 4.6 with $\nu(t) \equiv t$, we could analogously to Theorem 5.1 derive the criterion (1.7).

Now consider the case where (5.1) is violated.

Theorem 5.2. *Let $\alpha \in]0,1], \tau(t) \geq \alpha t$ for large t,*

$$\liminf_{t\to+\infty} t \int_t^{+\infty} p(s)\,ds = c_0 \in]0, c(\alpha)] \tag{5.11}$$

and

$$\limsup_{t\to+\infty} \left(\alpha t + \lambda_0 \int_0^{\alpha t} sp(s) \left(\int_t^{+\infty} p(\xi)\,d\xi\right)^{-1} ds\right) \int_t^{+\infty} p(s)\,ds > 1, \tag{5.12}$$

where $c(\alpha)$ is defined by (5.2) and λ_0 is the smaller root of the equation

$$\alpha^{\lambda-1}\lambda(1-\lambda) = c_0. \tag{5.13}$$

Then (1.1) is oscillatory.

Proof. In view of (5.11)–(5.13) one can choose $c^* \in]0, c_0[$ close enough to c_0, $\varepsilon > 0$ small enough and $t_0 \geq 0$ large enough for the inequalities

$$t \int_t^{+\infty} p(s)\,ds \geq c^* \quad \text{for} \quad t \geq t_0$$

and

$$\limsup_{t\to+\infty} \left(\alpha t + (\lambda^* - \varepsilon) \int_0^{\alpha t} sp(s) \left(\int_t^{+\infty} p(\xi)\,d\xi\right)^{-1} ds\right) \int_t^{+\infty} p(s)\,ds > 1 \tag{5.14}$$

to hold, where λ^* is the smaller root of $\alpha^{\lambda-1}\lambda(1-\lambda) = c^*$.

Let w be a solution of (2.11) with $\nu(t) \equiv \alpha t$. Defining λ_* by (5.5) and acting as in deriving the inequality (5.8), we get

$$\alpha^{\lambda_*-1}\lambda_*(1-\lambda_*) \geq c^*,$$

whence we get $\lambda_* \geq \lambda^*$. This means that

$$w(t) \geq (\lambda^* - \varepsilon) \left(\int_t^{+\infty} p(\xi)\,d\xi\right)^{-1} \quad \text{for large } t.$$

Therefore (5.14) and Corollary 4.6 imply that the equation (1.1) is oscillatory. The proof is complete.

In the class of the functions p satisfying

$$p(t) \geq \frac{c_0}{t^2} \quad \text{for large} \quad t, \tag{5.15}$$

we can get the following result which is similar to Theorem 3.4 in the sense that it connects the upper and lower limits of the same expression.

Theorem 5.3. *Let $\tau(t) \geq \alpha t$ for large t and (5.15) be fulfilled, where $\alpha \in]0,1]$, $c_0 \in]0, c(\alpha)]$ and $c(\alpha)$ is defined by (5.2). Let, moreover,*

$$\limsup_{t\to+\infty} t \int_t^{+\infty} p(s)\,ds > \frac{1}{\alpha(1+\lambda_0)}, \tag{5.16}$$

where λ_0 is the smaller root of (5.13). Then (1.1) is oscillatory.

Proof. Let $T \geq 0$ and w be a solution of (2.11) with $\nu(t) \equiv \alpha t$. Denote the left-hand side of (5.16) by p^*. By (5.16) there is a sufficiently small $\varepsilon > 0$ such that $p^*\alpha(1+\lambda_0-\varepsilon) > 1$. According to Corollary 4.6 it suffices to prove that

$$w(t) \geq \frac{\lambda_0 - \varepsilon}{c_0} t \quad \text{for large } t. \tag{5.17}$$

Denote $\beta_0 = \liminf_{t\to+\infty} w(t)/t$ (from (2.11) it follows that $\beta \geq \alpha > 0$), so for any $\beta \in]0, \beta_0[$ there is $t_0 \geq T$ such that $w(t) \geq \beta t$ for $t \geq t_0$. Suppose first that $\beta c_0 > 1$. Then (2.11) yields

$$\begin{aligned} w(t) &\geq \int_{t_0}^{\alpha t} \exp\left\{\int_s^t \frac{\beta c_0}{\xi} d\xi\right\} ds \geq \int_{t_0}^{\alpha t} \left(\frac{t}{s}\right)^{\beta c_0} ds = \\ &= \frac{\alpha^{1-\beta c_0} t}{\beta c_0 - 1}\left(\left(\frac{t_0}{\alpha t}\right)^{1-\beta c_0} - 1\right) \quad \text{for} \quad t \geq t_0/\alpha. \end{aligned} \tag{5.18}$$

So (5.17) is fulfilled. Analogously, if $\beta c_0 = 1$, then $w(t) \geq t\ln(\alpha t/t_0)$ for large t, and again (5.17) holds. Finally, let $\beta c_0 < 1$. Then, from (5.18), we get

$$\frac{w(t)}{t} \geq \frac{\alpha^{1-\beta c_0}}{1-\beta c_0}\left(1 - \left(\frac{t_0}{\alpha t}\right)^{1-\beta c_0}\right) \quad \text{for large } t.$$

Since $\beta \in]0, \beta_0[$ is arbitrary, passing to lower limit we obtain that $\lambda = \beta_0 c_0$ satisfies $\alpha^{\lambda-1}\lambda(1-\lambda) \geq c_0$. Hence $\beta_0 c_0 \geq \lambda_0$ which means that (5.17) is fulfilled. The proof is complete.

Finally we consider the case where the delay, roughly speaking, is like t^α.

Theorem 5.4. *Let $\alpha \in]0,1[$ and $\liminf_{t\to+\infty} \tau(t)t^{-\alpha} > 0$. Then the condition*

$$\liminf_{t\to+\infty} t^\alpha \int_t^{+\infty} p(s)\, ds > 0$$

is sufficient for (1.1) *to be oscillatory.*

Proof. The proof is quite analogous to that of Theorem 5.1, so it will be only sketched. Let $T \geq 0$, $\gamma > 0$ be such that $\tau(t) \geq \gamma t^\alpha$ for large t and w be a solution of (2.11) with $\nu(t) \equiv \gamma t^\alpha$. Define λ_* by (5.5) and suppose first that $\alpha\lambda_* \leq 1$. Let $\lambda < \lambda_*$ and $\beta > 0$ be such that $t^\alpha \int_t^{+\infty} p(s)\, ds \geq \beta$ for large t. Proceeding as in deriving (5.7), we obtain

$$w(t)\int_t^{+\infty} p(\zeta)\, d\zeta \geq \frac{\beta^\lambda \gamma^{1-\alpha\lambda}}{1-\alpha\lambda}\left(\int_t^{+\infty} p(\zeta)\, d\zeta\right)^{1-\lambda} t^{\alpha(1-\alpha\lambda)}\,[1+o(1)] \geq$$

$$\geq \frac{\beta^\lambda \gamma^{1-\alpha\lambda}}{1-\alpha\lambda}\left(t^\alpha \int_t^{+\infty} p(\zeta)\, d\zeta\right)^{1-\lambda} t^{\alpha\lambda(1-\alpha)}\,[1+o(1)] \geq Ct^{\alpha\lambda(1-\alpha)}\,[1+o(1)] \to +\infty$$

as $t \to +\infty$, where $C > 0$ is a constant. We were able to write the last inequality since like in (5.7) we can assume that $t^\alpha \int_t^{+\infty} p(s)\, ds \leq 1$ for large t and therefore the $(1-\lambda)$-th power can be estimated from below independently of whether $\lambda > 1$

or $\lambda \leq 1$. Thus we have $\lambda_* = +\infty$ which contradicts our assumption that $\alpha\lambda_* \leq 1$. Thus $\alpha\lambda_* > 1$ and we can take $\lambda \in]1/\alpha, \lambda_*[$ to get

$$w(t) \int_t^{+\infty} p(\zeta)\, d\zeta \geq \frac{\beta^\lambda \gamma^{1-\alpha\lambda}}{\alpha\lambda - 1} \left(\int_t^{+\infty} p(\zeta)\, d\zeta \right)^{1-\lambda} [C_0 + o(1)] \to +\infty$$

as $t \to +\infty$, where $C_0 > 0$ is a constant. Therefore $\lambda_* = +\infty$. Hence as in the proof of Theorem 5.1 we conclude that (5.9) holds for any $\lambda > 0$. Using this inequality with $\lambda = 2$ and writing down the chain of inequalities analogous to (5.9) (instead of $t^{1/2}$ one has to take $t^{\alpha/2}$), we can ascertain that the conditions of Corolllary 4.6 are fulfilled with $\nu(t) \equiv \gamma t^\alpha$. The proof is complete.

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